On the asymptotics of some large Hankel determinants generated by Fisher-Hartwig symbols defined on the real line

T. M. Garoni
Institute for Mathematics and its Applications, University of Minnesota,
400 Lind Hall, 207 Church Street S.E., Minneapolis, MN 55455-0436[*]
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We investigate the asymptotics of Hankel determinants of the form

$$\det_{j,k=0}^{N-1}\left[\int_{\Omega} dx\,\omega_N(x)\prod_{i=1}^{m}|\mu_i - x|^{2q_i}\,x^{j+k}\right]$$

as $N\to\infty$ with q and μ fixed, where Ω is an infinite subinterval of $\mathbb{R}$ and $\omega_N(x)$ is a positive weight on Ω. Such objects are natural analogues of Toeplitz determinants generated by Fisher-Hartwig symbols, and arise in random matrix theory in the investigation of certain expectations involving random characteristic polynomials. The reduced density matrices of certain one-dimensional systems of trapped impenetrable bosons can also be expressed in terms of Hankel determinants of this form.

We focus on the specific cases of scaled Hermite and Laguerre weights. We compute the asymptotics by using a duality formula expressing the $N\times N$ Hankel determinant as a $2(q_1+\cdots+q_m)$-fold integral, which is valid when each q_i is natural. We thus verify, for such q, a recent conjecture of Forrester and Frankel derived using a log-gas argument.

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I. INTRODUCTION

Consider the multiple integral

$$H_{M,N,m,q}(\mu) := \int_{\Omega}\omega_N(z_1)dz_1\ldots\int_{\Omega}\omega_N(z_M)dz_M\,|\Delta_M(z)|^2\prod_{l=1}^{M}\prod_{i=1}^{m}|\mu_i - z_l|^{2q_i}, \tag{1}$$

where

$$q := (q_1,\ldots,q_m), \qquad\qquad \mu := (\mu_1,\ldots,\mu_m), \tag{2}$$

$\omega_N(z)$ is a nonzero and continuous weight function, possibly depending on a parameter N, and

$$\Delta_M(z) := \det_{j,k=1}^{M}(z_k^{j-1}) = \prod_{1\le j<k\le M}(z_k - z_j) \tag{3}$$

is the Vandermonde determinant. As a notational convenience we also define

$$\begin{aligned} H_{M,N} &:= H_{M,N,0,\cdot}(\cdot), &(4)\\ &= \int_{\Omega}\omega_N(z_1)dz_1\ldots\int_{\Omega}\omega_N(z_M)dz_M\,|\Delta_M(z)|^2. &(5)\end{aligned}$$

It is known that M-fold integrals of the form (1) can be identified with the determinant of an $M\times M$ matrix. Expanding the Vandermonde determinants in terms of sums over permutations, and simplifying appropriately, we find

$$H_{M,N,m,q}(\mu) = M!\,\det_{j,k=0}^{M-1}\left[\int_{\Omega}dz\,a(z)\,z^j\,(z^*)^k\right], \tag{6}$$

where z^* denotes the complex conjugate of z, and

$$a(z) := \omega_N(z)\prod_{i=1}^{m}|\mu_i - z|^{2q_i}. \tag{7}$$

When $\Omega \subseteq \mathbb{R}$, equation (6) becomes

$$H_{M,N,m,q}(\mu) = M! \det_{j,k=0}^{M-1} \left[a_{j+k}\right], \tag{8}$$

where

$$a_n := \int_\Omega dz\, a(z)\; z^n. \tag{9}$$

One says that the determinant (8) is *generated* by the function (7).

Since the entries in the determinant (8) are of the form a_{j+k}, we have thus identified the multiple integral (1), when $\Omega \subseteq \mathbb{R}$, with a Hankel determinant. Had we instead taken Ω to be $\mathbb{T}$, the unit circle in $\mathbb{C}$, then according to (6) the entries of the determinant would be a_{j-k}, and we would thus obtain a Toeplitz determinant. (It is conventional when discussing such Toeplitz determinants to set $z = e^{i\theta}$ and to define a_n as the integral of $a(z)\, z^n$ with respect to $d\theta$ rather than dz. This merely introduces the nonzero factor $i\, e^{i\theta}$ which is easily absorbed into the definition of $a(z)$, and so such technicalities are not relevant to our discussion here.)

When $z \in \mathbb{T}$, functions of the form (7) are known as *Fisher-Hartwig symbols*[1] (although we remark that they are not the most general examples of Fisher-Hartwig symbols). By extension, we can describe the Hankel determinant (8) as being generated by a Fisher-Hartwig symbol which is defined on the real line.

The asymptotic analysis of Toeplitz determinants generated by Fisher-Hartwig symbols is a fascinating and well studied subject (see e.g.[1] and references therein), and rigorous results which describe the large M asymptotic behavior of Toeplitz determinants generated by symbols of the form (7) are known[2]. There are a number of important physical applications of such determinants (see e.g.[3,4,5]). It is often the case, as we discuss presently, that the quantity appearing in applications is actually the integral (1) rather than the determinant directly, and when $\Omega \subseteq \mathbb{R}$ we are naturally lead to Hankel determinants generated by the symbol (7). We discuss below a number of physical applications in which the asymptotics of such Hankel determinants is of interest. A rigorous treatment of these asymptotics is an open problem, however a conjectured form for the large M asymptotics has recently been reported by Forrester and Frankel in[5]. Very recently, the $m = 1$ case of this conjecture has been verified when $\Omega = \mathbb{R}$ and $\omega_N(x)$ is a Hermite weight, by using a Riemann Hilbert approach[6]. In the present work we verify the conjecture of Forrester and Frankel when $\omega_N(z)$ is either a Hermite or Laguerre weight for any $m \in \mathbb{N}$, when each $q_i \in \mathbb{N}$.

A. Random matrix theory

The multiple integral (1) has a natural interpretation in random matrix theory. Let us consider the ensemble of random matrices with joint eigenvalue probability density function (pdf) given by

$$P_{M,N}(x) := \frac{1}{H_{M,N}}\, |\Delta_M(x)|^2 \prod_{l=1}^{M} \omega_N(x_l), \tag{10}$$

and whose eigenvalues lie in Ω. When $\Omega = \mathbb{R}$, concrete examples of such ensembles include the ubiquitous Gaussian unitary ensemble (GUE), corresponding to $\omega_N(x) = \exp(-2Nx^2)$, as well as more general unitary ensembles (UE), corresponding to $\omega_N(x) = \exp(-NV(x))$ with $V(x)$ an arbitrary polynomial of even degree with positive leading coefficient (see e.g.[7]). When $\Omega = (0, \infty)$ an important example is the Laguerre unitary ensemble (LUE), corresponding to $\omega_N(x) = x^\alpha \exp(-4Nx)$, which includes Wishart matrices and the Chiral GUE as special cases (the latter after a straightforward change of variables; see e.g.[8]). Setting $x = e^{i\theta}$ and $\omega_N(x) = 1$ in (10) we obtain the joint pdf for the eigenphases $\theta_l \in [0, 2\pi]$ of the ensemble of random unitary matrices with Haar measure, often called the circular unitary ensemble (CUE). For the purpose of computing expectations, the CUE is equivalent to (10) with $\Omega = \mathbb{T}$ and $\omega_N(z) = 1/i\, z$.

If we denote the characteristic polynomial of the $M \times M$ matrix X, with eigenvalues $x_1, \ldots, x_M$, by

$$\mathcal{Z}_M(\mu_i) := \det(\mu_i\, I - X) = \prod_{l=1}^{M} (\mu_i - x_l), \tag{11}$$

then (1) corresponds to the following expectation involving the absolute value of such characteristic polynomials

$$\left\langle \prod_{i=1}^{m} |\mathcal{Z}_M(\mu_i)|^{2q_i} \right\rangle_{P_{M,N}} = \frac{H_{M,N,m,q}(\mu)}{H_{M,N}}, \tag{12}$$

where the expectation on the left hand side of (12) is with respect to the joint eigenvalue pdf (10). The case $M = N$ is generally the case of interest.

From (6) we see that expectations of characteristic polynomials of the form appearing in (12) are characterized by a determinant generated by the symbol (7); when $\Omega \subseteq \mathbb{R}$, it is a Hankel determinant, and when $\Omega = \mathbb{T}$ it is a Toeplitz determinant.

A sizable literature on the correlations of products and ratios of characteristic polynomials of random matrices from various ensembles has emerged in recent years, see e.g.[9,10,11,12,13,14,15,16], and significant progress has been made in the understanding of such objects. Such quantities have applications in diverse fields including number theory, quantum chaos and many-body quantum mechanics. These works consider either exact algebraic relations that are valid for finite N, or the large N asymptotics in the usual universal microscopic scaling limits. We shall be interested not in scaling limits, but in the limit of large N with μ fixed. Investigations of objects of the form (12) in this limit have been reported in[5,17].

B. Impenetrable bosons

A compelling physical motivation for investigating multiple integrals of the form (1) arises from a consideration of certain one-dimensional many-body systems of impenetrable bosons. By impenetrability we simply mean that we require the wavefunction to vanish whenever two bosons occupy the same point in space. Such systems have been receiving renewed theoretical interest recently due to the possibility of their experimental realization in the near future using ultra-cold systems of atomic bosons confined in elongated traps; see e.g.[18,19]. Systems of impenetrable bosons with certain specific boundary conditions are known to have ground state wavefunctions of the form

$$\psi(x_1,\ldots,x_M) = \frac{1}{\sqrt{C_M}} \prod_{l=1}^{M} \sqrt{g_N(x_l)}\ |\Delta_M[f(x_1),\ldots,f(x_M)]|, \qquad x_1,\ldots,x_M \in D \subseteq \mathbb{R}, \tag{13}$$

see e.g.[20,21]. Specifically, systems with periodic, Dirichlet, or Neumann boundary conditions have wavefunctions of this form, as do systems confined in an harmonic well. Of these, the harmonically confined system is perhaps the most relevant to current experiments.

It is worth emphasizing that the introduction of such zero-range infinite-strength interactions establishes a correspondence between impenetrable bosons and a corresponding system of free fermions, a fact first noted in[22], and this is one of the primary reasons for current experimental interest in such systems. Indeed, for the specific systems mentioned above, were it not for the absolute value surrounding the Vandermonde determinant, (13) would define the wavefunction for a system of free fermions. This fact implies that certain quantities such as the energy spectrum and the particle density are identical in the impenetrable boson system and its corresponding free fermion system. Quantities which depend on the phase of the wavefunction however will clearly differ significantly between these two systems.

One such quantity, of great significance, is the n-body density matrix, which for a system of $N+n$ particles is defined as

$$\rho_{N+n}^{(n)}(x_1,y_1,\ldots,x_n,y_n) := \binom{N+n}{n} \int_D d\xi_1 \ldots \int_D d\xi_N\, \psi(x_1,\ldots,x_n,\xi_1,\ldots,\xi_N)\psi^*(y_1,\ldots,y_n,\xi_1,\ldots,\xi_N). \tag{14}$$

A key observation is that wavefunctions of the form (13) admit the factorization

$$\psi(x_1,\ldots,x_n,\xi_1,\ldots,\xi_N) = \sqrt{\frac{C_N}{C_{N+n}}} \prod_{l=1}^{n} \sqrt{g_N(x_l)}\ |\Delta_n[f(x_1),\ldots,f(x_n)]| \prod_{l=1}^{N}\prod_{i=1}^{n} |f(x_i)-f(\xi_l)| \cdot \psi(\xi_1,\ldots,\xi_N), \tag{15}$$

and inserting (15) into (14) yields

$$\begin{aligned}\rho_{N+n}^{(n)}(x_1,y_1,\ldots,x_n,y_n) = \binom{N+n}{N} \prod_{i=1}^{n} \sqrt{g_N(x_i)\,g_N(y_i)}|\Delta_n[f(x_1),\ldots,f(x_n)]\Delta_n[f(y_1),\ldots,f(y_n)]| \\ \times \frac{1}{H_{N+n,N}} H_{N,N,2n,q}(f(x_1),\ldots,f(x_n),f(y_1),\ldots,f(y_n))\Big|_{q=(1/2,\ldots,1/2)},\end{aligned} \tag{16}$$

where in the definition of $H_{N,N,2n,q}$ and $H_{N+n,N}$ we have $\Omega = f(D)$ and

$$\omega_N(z) = g_N(f^{-1}(z))\,\frac{df^{-1}}{dz}(z). \tag{17}$$

For the four specific systems mentioned below (13), this $\omega_N(z)$ given in (17) is well defined, nonzero and continuous.

Hence, from (6) we see that $\rho^{(n)}_{N+n}$ is characterized by a determinant generated by the symbol (7). For systems subject to periodic boundary conditions this determinant will be a Toeplitz determinant[23,24], whereas for systems confined by an harmonic well it will be a Hankel determinant[24]. Indeed, this link between density matrices for impenetrable bosons and Toeplitz determinants generated by Fisher-Hartwig symbols was originally one of the key motivations for investigating the asymptotics of such Toeplitz determinants[3,4,23]. In light of the possible future experimental realization of finite one-dimensional harmonically trapped systems of impenetrable bosons, an important theoretical question is the behavior of the corresponding density matrices when N is large and $x_1, y_1, \ldots, x_n, y_n$ are fixed. This then provides a direct physical motivation for investigating the large N asymptotics of $N \times N$ Hankel determinants generated by symbols of the form (7).

Perhaps the most important quantity is the one-body density matrix. The asymptotics of the one-body density matrix for a system with periodic boundary conditions can be rigorously established from the asymptotics of the corresponding Toeplitz determinant. The leading order behavior of the one-body density matrix in the case of harmonic confinement was deduced in[24] using log-gas arguments, and has been recovered in[25] using a rather more direct, yet still non-rigorous, approach. The asymptotics of the one-body density matrix in the Dirichlet/Neumann case was deduced in[17], again by log-gas arguments, and has now been rigorously proved in[5], by making use of recent results in[26,27].

We conclude our discussion of impenetrable bosons by noting the correspondence between the joint eigenvalue pdf (10) and the wavefunction (13). The correspondence between the joint eigenvalue pdf (10) and the wavefunction for a system of free fermions is well known[28]. The correspondence between impenetrable bosons and random matrices was first noted by Sutherland[29], between systems of impenetrable bosons with periodic boundary conditions and the CUE; see also[30]. The correspondence between impenetrable bosons with Dirichlet or Neumann boundary conditions and the Jacobi unitary ensembles (JUE) was discussed in[17,20], and a similar interpretation for the LUE was noted in[5]. Again, the most interesting case from an experimental perspective is the correspondence between the GUE and systems of impenetrable bosons confined in an harmonic well, and this particular system has been the focus of considerable recent theoretical study, see e.g.[24,25,31] and references therein.

C. Asymptotics of Hankel determinants

The asymptotics of large Toeplitz and Hankel determinants has been of long standing interest to mathematicians. For Toeplitz determinants generated by well behaved symbols, very precise asymptotic results are given by the Szegö limit theorems (see e.g.[1,32]). Toeplitz determinants generated by the symbol (7) are not amenable to the Szegö limit theorems however since (7) has zeros. Inspired in part by applications to impenetrable bosons Lenard[4,23] (see also[3]) conjectured the asymptotics of Toeplitz determinants generated by symbols of the form (7), and this conjecture was subsequently proved by Widom[2]. The asymptotic behavior of Toeplitz determinants generated by (7), as well as more general Fisher-Hartwig symbols, is now well understood (see e.g.[1]). Analogously, the asymptotic behavior of large Hankel determinants generated by functions defined on $\Omega \subseteq \mathbb{R}$ has also been the subject of study. This problem was addressed by Szegö[33] and also Hirschman[34] with Ω a finite interval (see also[32]). In the context of the UE and LUE of random matrix theory, as well as in the context of trapped systems of impenetrable bosons, we are interested in case where Ω is infinite, and recently Basor *et al*[35] have considered the asymptotics of Hankel determinants generated by symbols defined on $\Omega = (0, \infty)$. However, a key restriction in these works is that the symbol be nowhere zero, and hence they do not apply to determinants generated by (7). Forrester and Frankel[5] have recently conjectured the asymptotic behavior of Hankel determinants generated by symbols of the form (7) defined on $\Omega \subseteq \mathbb{R}$. Complete and rigorous proofs of their conjectures remains an open problem. As mentioned above, a rigorous proof for the $m = 1$ case when $\Omega = \mathbb{R}$ and $\omega_N(x)$ is a Hermite weight has very recently been reported in[6].

Specifically, Forrester and Frankel[5] consider the behavior of the ratio

$$\mathcal{H}_{N,m,q}(\mu) := \frac{H_{N,N,m,q}(\mu)}{H_{N+|q|,N}}, \tag{18}$$

where

$$|q| := q_1 + \cdots + q_m, \tag{19}$$

as $N \to \infty$ with μ and q fixed. Amongst other results, they consider the case $\omega_N(x) = e^{-N V(x)}$ with either $\Omega = \mathbb{R}$ or $\Omega = (0, \infty)$. For a particular choice of such $\omega_N(x)$ and Ω there corresponds the quantity $\rho(x)$, which, with $P_{N,N}$

defined as in (10), equals the limit of

$$\int_{\Omega^{N-1}} P_{N,N}(x, x_2, \ldots, x_N)\, dx_2 \ldots dx_N \tag{20}$$

as $N \to \infty$ with x held fixed; i.e. $\rho(x)$ is the limiting expected eigenvalue density of the ensemble of random matrices defined by $P_{N,N}$. We note that $\rho(x)$ is non-negative and has compact support. In what follows $\mathrm{int}(\mathrm{supp}\,\rho)$ denotes the interior of the support of $\rho(x)$. For a detailed discussion of $\rho(x)$, and a number of other very interesting alternative characterizations of $\rho(x)$, the reader is referred to[7]. The conjecture reported in[5] (in our notation) is the following.

Conjecture (Forrester-Frankel). *Let $m, N \in \mathbb{N}$, $q \in (-1/2, \infty)^m$ and $\mu_1, \ldots, \mu_m \in \mathrm{int}(\mathrm{supp}\ \rho)$, where $\rho(x)$ is defined as above. Furthermore, suppose either $\Omega = \mathbb{R}$ or $\Omega = (0, \infty)$, and take $\omega_N(x) = e^{-N\,V(x)}$ where $V(x)$ is a polynomial which is independent of N and which has positive leading coefficient and no zeros in Ω. Then*

$$\begin{aligned} \mathcal{H}_{N,m,q}(\mu) = N^{\sum_{i=1}^m (q_i^2 - q_i)} \prod_{i=1}^m [\omega_N(\mu_i)]^{-q_i} \prod_{i=1}^m \frac{G^2(q_i+1)}{G(2q_i+1)} (2\pi)^{q_i^2 - q_i} \\ \times \prod_{1 \le j < k \le m} |\mu_k - \mu_j|^{-2q_j q_k} \prod_{i=1}^m [\rho(\mu_i)]^{q_i^2}\ [1 + o(1)]\,. \end{aligned} \tag{21}$$

Here and in the sequel G refers to Barnes' G-function[36].

Despite first appearances, the structure of (21) is actually quite simple. Note that the only ensemble dependent quantities on the right hand side are $\rho(x)$ and $\omega_N(x)$, and that the dependence of $\mathcal{H}_{M,m,q}(\mu)$ on these two quantities is universal. The other quantities on the right hand side are truly universal. We note that the factor $G^2(q_i+1)/G(2q_i+1)$ occurs also in the asymptotics of Toeplitz determinants generated by Fisher-Hartwig symbols[2], and has been discussed in the context of moments of random characteristic polynomials[9,10]. We should note that the actual conjecture reported in[5] is slightly more general than (21), but (21) is sufficient for our purposes.

As an aside, we remark that the ratio appearing in (16) is precisely of the form (18) and so the asymptotic behavior of all the n-body density matrices for a system of harmonically confined impenetrable bosons follows directly from (21).

The present article focuses on two classical cases already mentioned, the case of the Hermite weight, corresponding to the GUE, and the case of the Laguerre weight, corresponding to the LUE. In the Hermite case

$$\omega_N(x) = e^{-2Nx^2}, \qquad \Omega = \mathbb{R}, \qquad \rho(\mu) = \frac{2}{\pi}\sqrt{1-\mu^2}, \qquad \mathrm{supp}(\rho) = [-1, 1], \tag{22}$$

and in the Laguerre case

$$\omega_N(x) = x^\alpha\, e^{-4Nx}, \qquad \Omega = (0, \infty), \qquad \rho(\mu) = \frac{2}{\pi}\sqrt{\frac{1}{\mu} - 1}, \qquad \mathrm{supp}(\rho) = [0, 1]. \tag{23}$$

The conjecture reported in[5] was deduced by considering the specific examples of the Hermite and Laguerre cases, to find the general form in terms of $\rho(x)$ and $\omega_N(x)$. The asymptotics for these two cases was deduced by using the log-gas analogy to conjecture a factorization of (18), computing the asymptotics of each factor when the q_i were natural, and then conjecturing an analytic continuation to real q_i.

In this work we show how (21) can be proved rigorously for the Hermite and Laguerre cases, when each q_i is natural, by using a duality formula derived from a general result in[9]. By a *duality* formula we mean an equation identifying the N-fold integral $\mathcal{H}_{N,m,q}(\mu)$ with a $2|q|$ - fold integral. Our results clarify the origin of the factors appearing in (21). A similar approach has been used in[25] to investigate a particular special case when the weight was of Hermite type, in the context of trapped impenetrable bosons.

Section II contains a discussion of the duality formula. Due to the similar nature of the Laguerre and Hermite cases, they can both be derived simultaneously. Section III then discusses the asymptotic analysis of the $2|q|$ - fold integral obtained from the duality formula by means of the saddle point method. We show how to deduce the general form of all terms and explicitly simplify the leading order term, and thus verify the conjecture (21) for the Hermite and Laguerre cases when each q_i is natural.

II. DUALITY FORMULA

Let us define

$$F_{K,N}(\lambda_1,\ldots,\lambda_K) := \frac{1}{\Delta_K(\lambda_1,\ldots,\lambda_K)} \det_{j,k=1}^{K} \pi^{(N)}_{N+j-1}(\lambda_k), \tag{24}$$

where $\{\pi^{(N)}_k\}_{k=0}^{\infty}$ are the monic orthogonal polynomials corresponding to $\omega_N(x)$ and Ω; i.e. they are uniquely defined by the following two conditions:

$$\int_\Omega \omega_N(x)\,dx\ \pi^{(N)}_j(x)\pi^{(N)}_k(x) = 0, \qquad j \neq k \tag{25}$$

$$\pi^{(N)}_j(x) = x^j + O(x^{j-1}). \tag{26}$$

For the Hermite and Laguerre cases, the $\pi^{(N)}_{N+j-1}(x)$ can be expressed in terms of the standard Hermite and Laguerre polynomials found in Szegö's classic book[37] as follows

$$\pi^{(N)}_{N+j-1}(x) = \begin{cases} 2^{-3(N+j-1)/2} N^{-(N+j-1)/2} H_{N+j-1}(\sqrt{2N}x), & \text{Hermite,} \\ (-1)^{N+j-1}(N+j-1)!(4N)^{-N-j+1}\, L^{(\alpha)}_{N+j-1}(4Nx), & \text{Laguerre.} \end{cases} \tag{27}$$

According to Brézin and Hikami[9], we have the following very useful identity

$$F_{K,N}(\lambda_1,\ldots,\lambda_K) = \frac{1}{H_{N,N}} \int_\Omega \omega_N(x_1)dx_1 \ldots \int_\Omega \omega_N(x_N)dx_N\ \Delta^2_N(x) \prod_{l=1}^{N}\prod_{i=1}^{K}(\lambda_i - x_l). \tag{28}$$

If we restrict ourselves to $q \in \mathbb{N}^m$ and set

$$K = 2q_1 + \cdots + 2q_m = 2|q|, \tag{29}$$

we can consider the confluent limit

$$\lim_{*} := \lim_{\lambda_{\tau(m)+2q_m}\to\mu_m} \cdots \lim_{\lambda_{\tau(m)+1}\to\mu_m} \cdots \lim_{\lambda_{2q_1+2q_2}\to\mu_2} \cdots \lim_{\lambda_{2q_1+1}\to\mu_2} \lim_{\lambda_{2q_1}\to\mu_1} \cdots \lim_{\lambda_1\to\mu_1}, \tag{30}$$

where

$$\tau(i) := \sum_{l=1}^{i-1} 2q_l. \tag{31}$$

Taking the limit (30) of both sides of (28), and using the elementary fact that $(\mu_i - x_l)^{2q_i} = |\mu_i - x_l|^{2q_i}$ when q_i is an integer and μ_i and x_l are real, we thus obtain

$$\lim_{*} F_{K,N}(\lambda_1,\ldots,\lambda_K) = \frac{H_{N,N,m,q}(\mu)}{H_{N,N}}, \tag{32}$$

and therefore

$$\mathcal{H}_{N,m,q}(\mu) = \frac{H_{N,N}}{H_{N+|q|,N}}\ \lim_{*} F_{K,N}(\lambda_1,\ldots,\lambda_K), \qquad\qquad q \in \mathbb{N}^m. \tag{33}$$

This is the key relation we need to derive the duality formula for $\mathcal{H}_{N,m,q}(\mu)$, all that remains is to take the confluent limit of $F_{K,N}$.

For later convenience, we set

$$\pi^{(N)}_{N+j-1}(x) = \zeta_N(x)\, r_{N+j-1}(x). \tag{34}$$

If we insert (34) into (24) and take the limit (30), then by factoring the Vandermonde determinant we obtain

$$\lim_{*} F_{K,N}(\lambda_1,\ldots,\lambda_K) = \lim_{*} \prod_{1\le j<k\le m} \prod_{l_k=1}^{2q_k} \prod_{l_j=1}^{2q_j} (\lambda_{\tau(k)+l_k} - \lambda_{\tau(j)+l_j})^{-1} \prod_{j=1}^{K} \zeta_N(\lambda_j)$$
$$\times \lim_{*} \prod_{i=1}^{m} \Delta_{2q_i}^{-1}(\lambda_{\tau(i)+1},\ldots,\lambda_{\tau(i+1)}) \det_{j,k=1}^{K} r_{N+j-1}(\lambda_k), \tag{35}$$
$$= \prod_{1\le j<k\le m} (\mu_k-\mu_j)^{-4q_kq_j} \prod_{i=1}^{m} \zeta_N^{2q_i}(\mu_i)$$
$$\times \lim_{*} \prod_{i=1}^{m} \Delta_{2q_i}^{-1}(\lambda_{\tau(i)+1},\ldots,\lambda_{\tau(i+1)}) \det_{j,k=1}^{K} r_{N+j-1}(\lambda_k). \tag{36}$$

To compute the remaining limit in (36) we can use the following.

Lemma 1. *Let* $\mathbf{c}(\lambda)$ *denote a column vector with* $\tau(m+1)$ *entries, then for* $i=1,2,\ldots,m$

$$\lim_{\lambda_{\tau(i)+2q_i}\to\mu_i} \cdots \lim_{\lambda_{\tau(i)+1}\to\mu_i} \frac{1}{\Delta_{2q_i}(\lambda_{\tau(i)+1},\ldots,\lambda_{\tau(i)+2q_i})}$$
$$\times G(2q_i+1)\det\left[\mathbf{c}(\lambda_1)\ldots\mathbf{c}(\lambda_{\tau(i)+1})\ldots\mathbf{c}(\lambda_{\tau(i)+2q_i})\ldots\mathbf{c}(\lambda_{\tau(m+1)})\right]$$
$$= \det\left[\mathbf{c}(\lambda_1)\ldots\mathbf{c}(\lambda_{\tau(i)})\ \mathbf{c}(\mu_i)\ \frac{d}{d\mu_i}\mathbf{c}(\mu_i)\ldots\frac{d^{2q_i-1}}{d\mu_i^{2q_i-1}}\mathbf{c}(\mu_i)\ \mathbf{c}(\lambda_{\tau(i+1)+1})\ldots\mathbf{c}(\lambda_{\tau(m+1)})\right], \tag{37}$$

where G *is Barnes' G-function.*

Proof. This is easily proven by induction using L'Hôpital's rule, and recalling the identity

$$\prod_{l=1}^{n}\Gamma(l) = G(n+2). \tag{38}$$

□

Applying Lemma 1 to (36) independently for each set $\{\lambda_{\tau(i)+1},\ldots,\lambda_{\tau(i)+2q_i}\}$ with $i=1,2,\ldots,m$, results in

$$\lim_{*} F_{K,N}(\lambda_1,\ldots,\lambda_K) = \prod_{1\le j<k\le m} (\mu_k-\mu_j)^{-4q_kq_j} \prod_{i=1}^{m} \frac{\zeta_N^{2q_i}(\mu_i)}{G(2q_i+1)} \det_{\substack{1\le l_i\le 2q_i\\ 1\le i\le m\\ 1\le j\le 2|q|}} \left[\frac{d^{l_i-1}}{d\mu_i^{l_i-1}} r_{N+j-1}(\mu_i)\right]. \tag{39}$$

In (39) the columns of the determinant are ordered such that one starts with $i=1$, writes out the $2q_1$ columns depending on l_1, and then moves to $i=2$ etc. We remark that we have now already obtained one of the two Barnes G-function factors that appear in (21).

The special property possessed by the Hermite and Laguerre polynomials that allows us to derive a duality formula for $\mathcal{H}_{N,m,q}(\mu)$ for the specific weights (22) and (23) is that they can be expressed in terms of contour integrals. Indeed, by suitably massaging the standard results in Szegö's book[37] we find

$$\pi_{N+j-1}^{(N)}(\mu_i) = \begin{cases} c_j(N)\, e^{2N\mu_i^2} \displaystyle\int_{\mathcal{C}} dz\, e^{-2Nz\,\mu_i + Nz^2/2} z^{N+j-1} & \text{Hermite,} \\ c_j(N) \displaystyle\int_{\mathcal{C}} dz\, e^{-2Nz\,\mu_i} \frac{(z+2)^{N+\alpha}}{z^{N+1}} \left(\frac{1}{z}+\frac{1}{2}\right)^{j-1} & \text{Laguerre,} \end{cases} \tag{40}$$

$$c_j(N) = \begin{cases} \sqrt{\dfrac{2N}{\pi}}\, \dfrac{1}{i\, 2^{N+j}} & \text{Hermite,} \\ (-1)^{N+j-1} \dfrac{(N+j-1)!}{N^{N+j-1}} \dfrac{1}{2^{2N+j+\alpha}\pi i} & \text{Laguerre,} \end{cases} \tag{41}$$

where in the Hermite case the contour $\mathcal{C}$ lies along the imaginary axis and is oriented from $-i\infty$ to $+i\infty$, and in the Laguerre case $\mathcal{C}$ is a closed positively oriented contour which encircles the origin but does not contain the point $z=-2$.

It is now straightforward to compute the derivatives required in (39) from the contour integrals in (40). Defining

$$\zeta_N(\mu_i) = \begin{cases} e^{2N\,\mu_i^2} & \text{Hermite,} \\ 1 & \text{Laguerre,} \end{cases} \tag{42}$$

and recalling the definition (34) we obtain

$$\frac{d^{l_i-1}}{d\mu_i^{l_i-1}} r_{N+j-1}(\mu_i) = c_j(N)\, d_{l_i}(N) \int_{\mathcal{C}} dz\; e^{-NS(z,\mu_i)}\, u(z)\, z^{l_i-1}[z^\delta + c]^{j-1}, \tag{43}$$

where

$$S(z,\mu_i) := \begin{cases} 2\mu_i\, z - \log(z) - \dfrac{z^2}{2} & \text{Hermite,} \\ 2\mu_i\, z + \log(z) - \log(z+2) & \text{Laguerre,} \end{cases} \tag{44}$$

$$u(z) := \begin{cases} 1 & \text{Hermite,} \\ \dfrac{(z+2)^\alpha}{z} & \text{Laguerre,} \end{cases} \tag{45}$$

$$d_{l_i}(N) := (-2N)^{l_i-1}, \tag{46}$$

and where $\delta = \pm 1$ and $c = 0, 1/2$ in the Hermite and Laguerre cases respectively.

Our task now is to simplify the determinant appearing in (39) by using the contour integral (43). This is achieved by the following lemma.

Lemma 2. *If $q \in \mathbb{N}^m$ and $\delta = \pm 1$, and we define*

$$b_{\delta,q_i}(z) := \begin{cases} 1 & \delta = +1, \\ i\, z^{1-2q_i} & \delta = -1, \end{cases} \tag{47}$$

then

$$\begin{aligned} &\det_{\substack{1\le l_i\le 2q_i \\ 1\le i\le m \\ 1\le j\le 2|q|}} \left[\int_{\mathcal{C}} dz_{\tau(i)+l_i}\; e^{-NS(z_{\tau(i)+l_i},\mu_i)}\, u(z_{\tau(i)+l_i})\, z_{\tau(i)+l_i}^{l_i-1} [z_{\tau(i)+l_i}^\delta + c]^{j-1} \right] \\ &= \prod_{i=1}^m \frac{1}{\Gamma(2q_i+1)} \prod_{i=1}^m \prod_{l_i=\tau(i)+1}^{\tau(i+1)} \int_{\mathcal{C}} dz_{l_i}\, e^{-NS(z_{l_i},\mu_i)}\, u(z_{l_i})\, b_{\delta,q_i}(z_{l_i}) \prod_{i=1}^m \Delta_{2q_i}^2(z_{\tau(i)+1},\ldots,z_{\tau(i+1)}) \\ &\qquad \times \prod_{1\le j<k\le m} \prod_{l_k=\tau(k)+1}^{\tau(k+1)} \prod_{l_j=\tau(j)+1}^{\tau(j+1)} (z_{l_k}^\delta - z_{l_j}^\delta). \end{aligned} \tag{48}$$

Proof. We start with the identity

$$\begin{aligned} &\det_{\substack{1\le l_i\le 2q_i \\ 1\le i\le m \\ 1\le j\le 2|q|}} \left[\int dz_{\tau(i)+l_i}\; g_{l_i}(z_{\tau(i)+l_i},\mu_i) [f(z_{\tau(i)+l_i})]^{j-1} \right] \\ &= \prod_{i=1}^m \prod_{l_i=1}^{2q_i} \int dz_{\tau(i)+l_i}\; g_{l_i}(z_{\tau(i)+l_i},\mu_i) \Delta_{2|q|}\left(f(z_1),\ldots,f(z_{2|q|})\right), \end{aligned} \tag{49}$$

which is valid for arbitrary integrable functions $f(z)$ and $g_{l_i}(z,\mu_i)$. If we apply (49) to the left hand side (LHS) of (48) and use the elementary fact that

$$\Delta_n(z_1+c, z_2+c, \ldots, z_n+c) = \Delta_n(z_1, z_2, \ldots, z_n), \tag{50}$$

we obtain

$$\text{LHS of (48)} = \prod_{i=1}^m \prod_{l_i=1}^{2q_i} \int_{\mathcal{C}} dz_{\tau(i)+l_i}\, z_{\tau(i)+l_i}^{l_i-1} \prod_{i=1}^m \prod_{l_i=\tau(i)+1}^{\tau(i+1)} e^{-NS(z_{l_i},\mu_i)}\, u(z_{l_i}) \cdot \Delta_{2|q|}(z_1^\delta,\ldots,z_{2|q|}^\delta). \tag{51}$$

To proceed further we first note the following two useful identities.

Lemma 3. *If $f(z_1,\ldots,z_{2|q|})$ is a totally antisymmetric function of each set of variables $\{z_{\tau(i)+1},\ldots,z_{\tau(i+1)}\}$, for $i=1,2,\ldots,m$, then*

$$\prod_{i=1}^{m}\prod_{l_i=1}^{2q_i}\int dz_{\tau(i)+l_i}\, z_{\tau(i)+l_i}^{l_i-1}\, f(z_1,\ldots,z_{2|q|})$$
$$=\prod_{i=1}^{m}\frac{1}{\Gamma(2q_i+1)}\prod_{i=1}^{m}\left(\prod_{l_i=\tau(i)+1}^{\tau(i+1)}\int dz_{l_i}\right)\Delta_{2q_i}(z_{\tau(i)+1},\ldots,z_{\tau(i+1)})\,.\,f(z_1,\ldots,z_{2|q|}) \tag{52}$$

Proof. By expanding the Vandermonde determinant and then rearranging the order of integrations we see that

$$\prod_{i=1}^{m}\left(\prod_{l_i=\tau(i)+1}^{\tau(i+1)}\int dz_{l_i}\right)\Delta_{2q_i}(z_{\tau(i)+1},\ldots,z_{\tau(i+1)})\,.\,f(\ldots,z_{\tau(i)+1},\ldots,z_{\tau(i)+2q_i},\ldots)$$
$$=\prod_{i=1}^{m}\sum_{\sigma_i\in S_{2q_i}}\prod_{l_i=1}^{2q_i}\int dz_{\tau(i)+\sigma_i(l_i)}\, z_{\tau(i)+\sigma_i(l_i)}^{l_i-1}\,(-1)^{\sigma_i}\,.\,f(\ldots,z_{\tau(i)+1},\ldots,z_{\tau(i)+2q_i},\ldots), \tag{53}$$

and the antisymmetry of f then implies that the right hand side of (53) equals

$$\prod_{i=1}^{m}\sum_{\sigma_i\in S_{2q_i}}\prod_{l_i=1}^{2q_i}\int dz_{\tau(i)+\sigma_i(l_i)}\, z_{\tau(i)+\sigma_i(l_i)}^{l_i-1}\,\cdot\, f(\ldots,z_{\tau(i)+\sigma_i(1)},\ldots,z_{\tau(i)+\sigma_i(2q_i)},\ldots)$$
$$=\prod_{i=1}^{m}\sum_{\sigma_i\in S_{2q_i}}\prod_{l_i=1}^{2q_i}\int dz_{\tau(i)+l_i}\, z_{\tau(i)+l_i}^{l_i-1}\,.\,f(\ldots,z_{\tau(i)+1},\ldots,z_{\tau(i)+2q_i},\ldots), \tag{54}$$

where the last equality follows by simply relabelling integration variables. The stated result is now immediate. □

Lemma 4. *With q, δ and $b_{\delta,q_i}(z)$ as defined in Lemma 2 we have*

$$\Delta_{2q_i}\left(z_1^{\delta},\ldots,z_{2q_i}^{\delta}\right)\Delta_{2q_i}\left(z_1,\ldots,z_{2q_i}\right)=\prod_{l=1}^{2q_i}b_{\delta,q_i}(z_l)\,\cdot\,\Delta_{2q_i}^2(z_1,\ldots,z_{2q_i}). \tag{55}$$

Proof. When $\delta=1$ there is nothing to prove, so take $\delta=-1$. Then

$$\Delta_{2q_i}\left(z_1^{\delta},\ldots,z_{2q_i}^{\delta}\right)\Delta_{2q_i}\left(z_1,\ldots,z_{2q_i}\right)=\prod_{1\le j<k\le 2q_i}\frac{(z_j-z_k)(z_k-z_j)}{z_jz_k}, \tag{56}$$
$$=(-1)^{2q_i^2+q_i}\prod_{1\le j<k\le 2q_i}\frac{1}{z_jz_k}\,\cdot\,\Delta_{2q_i}^2(z_1,\ldots,z_{2q_i}), \tag{57}$$
$$=\prod_{l=1}^{2q_i} i\, z_l^{1-2q_i}\,\cdot\,\Delta_{2q_i}^2(z_1,\ldots,z_{2q_i}). \tag{58}$$

□

Armed with Lemmas 3 and 4 the proof of Lemma 2 follows at once. Applying Lemma 3 to (51) results in

$$\text{LHS of (48)} = \prod_{i=1}^{m} \frac{1}{\Gamma(2q_i+1)} \prod_{i=1}^{m} \left(\prod_{l_i=\tau(i)+1}^{\tau(i+1)} \int_{\mathcal{C}} dz_{l_i} \right) \Delta_{2q_i}(z_{\tau(i)+1}, \ldots, z_{\tau(i+1)})$$
$$\times \prod_{i=1}^{m} \prod_{l_i=\tau(i)+1}^{\tau(i+1)} e^{-NS(z_{l_i},\mu_i)} u(z_{l_i}) \, . \, \Delta_{2|q|}(z_1^\delta, \ldots, z_{2|q|}^\delta) \tag{59}$$
$$= \prod_{i=1}^{m} \frac{1}{\Gamma(2q_i+1)} \prod_{i=1}^{m} \prod_{l_i=\tau(i)+1}^{\tau(i+1)} \int_{\mathcal{C}} dz_{l_i} e^{-NS(z_{l_i},\mu_i)} u(z_{l_i})$$
$$\times \prod_{i=1}^{m} \Delta_{2q_i}(z_{\tau(i)+1}^\delta, \ldots, z_{\tau(i+1)}^\delta) \Delta_{2q_i}(z_{\tau(i)+1}, \ldots, z_{\tau(i+1)})$$
$$\times \prod_{1\le j<k\le m} \prod_{l_k=\tau(k)+1}^{\tau(k+1)} \prod_{l_j=\tau(j)+1}^{\tau(j+1)} (z_{l_k}^\delta - z_{l_j}^\delta). \tag{60}$$

Applying Lemma 4 to the right hand side of (60) produces the stated result. □

Now we substitute (43) into (39), factor out the constants $c_j(N)$ and $d_{l_i}(N)$ from the determinant, and apply Lemma 2 to finally obtain

Proposition 1.

$$\mathcal{H}_{N,m,q}(\mu) = h_{N,m,q} \prod_{i=1}^{m} \frac{1}{G(2q_i+1)\Gamma(2q_i+1)} \prod_{i=1}^{m} \zeta_N^{2q_i}(\mu_i) \prod_{1\le j<k\le m} (\mu_k - \mu_j)^{-4q_j q_k} \cdot I_{N,m,q}(\mu), \tag{61}$$

where

$$I_{N,m,q}(\mu) := \prod_{i=1}^{m} \prod_{l_i=\tau(i)+1}^{\tau(i+1)} \int_{\mathcal{C}} dz_{l_i} \, e^{-NS(z_{l_i},\mu_i)} \, \Delta_{2q_i}^2(z_{\tau(i)+1}, \ldots, z_{\tau(i+1)})$$
$$\times \prod_{i=1}^{m} \prod_{l_i=\tau(i)+1}^{\tau(i+1)} g_{q_i}(z_{l_i}) \prod_{1\le j<k\le m} \prod_{l_k=\tau(k)+1}^{\tau(k+1)} \prod_{l_j=\tau(j)+1}^{\tau(j+1)} (z_{l_k}^\delta - z_{l_j}^\delta), \tag{62}$$

the function $g_{q_i}(z)$ *is*

$$g_{q_i}(z) := u(z)\, b_{\delta,q_i}(z) \tag{63}$$
$$= \begin{cases} 1 & \textit{Hermite,} \\ i\dfrac{(z+2)^\alpha}{z^{2q_i}} & \textit{Laguerre,} \end{cases} \tag{64}$$

and

$$h_{N,m,q} := \prod_{j=1}^{\tau(m+1)} c_j(N) \prod_{i=1}^{m} \prod_{l_i=1}^{2q_i} d_{l_i}(N) \cdot \frac{H_{N,N}}{H_{N+|q|,N}}. \tag{65}$$

Proposition 1 is an exact duality formula when $q \in \mathbb{N}^m$, expressing the N-fold integral $\mathcal{H}_{N,m,q}(\mu)$ in terms of the $2|q|$-fold integral $I_{N,m,q}(\mu)$. This allows us to compute the large N asymptotics of $\mathcal{H}_{N,m,q}(\mu)$ by computing the large N asymptotics of $I_{N,m,q}(\mu)$, and the latter can be obtained by using the saddle point method. This is the subject of Section III.

The prefactor $h_{N,m,q}$ defined in (65) can be expressed in terms of the Barnes' G-function by using known results for the Selberg Integral, see e.g.[8,28], and the asymptotics can then be obtained from the known asymptotics of Barnes'

G-function[38]. We obtain:

$$h_{N,m,q} = \begin{cases} 2^{-|q|^2-3|q|/2+\sum_{i=1}^m 2q_i^2}\pi^{-3|q|/2}\, N^{\sum_{i=1}^m 2q_i^2+|q|^2/2+|q|N}\dfrac{G(N+2)}{G(N+|q|+2)} & \text{Hermite,} \\ (-1)^{|q|}\, 2^{-2\,|q|+\sum_{i=1}^m 2\,{q_i}^2}\, \pi^{-2\,|q|}\, N^{(\alpha-|q|)\,|q|+\sum_{i=1}^m 2\,{q_i}^2} & \\ \quad\times\dfrac{G(N+2)}{G(N+1)}\dfrac{G(N+\alpha+1)}{G(N+\alpha+|q|+1)}\dfrac{G(N+2|q|+1)}{G(N+|q|+2)} & \text{Laguerre,} \end{cases} \tag{66}$$

$$= \begin{cases} N^{\sum_{i=1}^m(2q_i^2-q_i)}e^{|q|N}2^{-|q|^2}\displaystyle\prod_{i=1}^m 2^{2q_i^2-2q_i}\pi^{-2\,q_i}\left[1+O\left(\frac{1}{N}\right)\right] & \text{Hermite,} \\ N^{\sum_{i=1}^m(2q_i^2-q_i)}\displaystyle\prod_{i=1}^m(-1)^{q_i}\, 2^{2\,{q_i}^2-2\,q_i}\pi^{-2\,q_i}\left[1+O\left(\frac{1}{N}\right)\right] & \text{Laguerre.} \end{cases} \tag{67}$$

It will be useful in Section III for us to introduce the notation

$$h_{N,m,q} = N^{\sum_{i=1}^m(2q_i^2-q_i)}\, h_0\left[1+O\left(\frac{1}{N}\right)\right], \tag{68}$$

where

$$h_0 := \begin{cases} e^{|q|N}2^{-|q|^2}\displaystyle\prod_{i=1}^m 2^{2q_i^2-2q_i}\pi^{-2\,q_i} & \text{Hermite,} \\ \displaystyle\prod_{i=1}^m(-1)^{q_i}\, 2^{2\,{q_i}^2-2\,q_i}\pi^{-2\,q_i} & \text{Laguerre.} \end{cases} \tag{69}$$

III. ASYMPTOTICS

Now we begin the task of computing the large N asymptotics of the integral $I_{N,m,q}(\mu)$ for fixed μ and q. Since the only appearance that N makes in (62) is in the exponent of $e^{-N\,S(z,\mu_i)}$, this problem is a natural candidate for the saddle point method.

In both the Hermite and Laguerre cases, the function $S(z,\mu_i)$ has two saddle points, $z_{+,i}$ and its complex conjugate $z^*_{+,i}$. Explicitly

$$z_{+,i} = \begin{cases} \mu_i + i\sqrt{1-\mu_i^2} & \text{Hermite,} \\ -1+i\sqrt{\dfrac{1}{\mu_i}-1} & \text{Laguerre.} \end{cases} \tag{70}$$

It is worth noting that in both cases

$$\mathrm{Im}\{z_{+,i}\} = \frac{\pi}{2}\,\rho(\mu_i), \tag{71}$$

where $\rho(\mu_i)$ is as defined in (22) and (23), for the Hermite and Laguerre cases respectively. Both saddle points are of equal importance, since

$$\mathrm{Re}\{S(z_{+,i},\mu_i)\} = \mathrm{Re}\{S(z^*_{+,i},\mu_i)\}, \tag{72}$$

and we deform the contour $\mathcal{C}$ through both of them.

Let us denote the subset of the contour in neighborhoods of $z_{+,i}$ and $z^*_{+,i}$ by $\Omega_{+,i}$ and $\Omega_{-,i}$ respectively, and the complement of the union of these two neighborhoods in $\mathcal{C}$ by $\mathcal{C}_s$, so that

$$\mathcal{C} = \mathcal{C}_s \cup \Omega_{+,i} \cup \Omega_{-,i}. \tag{73}$$

By deforming $\mathcal{C}$ appropriately, the dominant contribution of each integral comes from $\Omega_{+,i}$ and $\Omega_{-,i}$, and the standard arguments of the saddle point method lead to

$$\begin{aligned} I_{N,m,q}(\mu) = \prod_{i=1}^m \prod_{l_i=\tau(i)+1}^{\tau(i+1)} \left(\int_{\Omega_{+,i}} dz_{l_i} + \int_{\Omega_{-,i}} dz_{l_i}\right) e^{-N\,S(z_{l_i},\mu_i)}\cdot\Delta^2_{2q_i}(z_{\tau(i)+1},\ldots,z_{\tau(i+1)}) \\ \times\prod_{i=1}^m \prod_{l_i=\tau(i)+1}^{\tau(i+1)} g_{q_i}(z_{l_i}) \prod_{1\le j<k\le m}\prod_{l_k=\tau(k)+1}^{\tau(k+1)}\prod_{l_j=\tau(j)+1}^{\tau(j+1)}(z^\delta_{l_k}-z^\delta_{l_j}) + \prod_{i=1}^m e^{-2q_i\,N\,\mathrm{Re}\{S_i\}}\cdot O(e^{-\varepsilon N}), \end{aligned} \tag{74}$$

for suitably small $\varepsilon > 0$, where we have defined

$$S_i := S(z_{+,i}, \mu_i) = S(z^*_{+,i}, \mu_i)^*. \tag{75}$$

We would now like to expand out the $2|q|$-fold composition of the sum of the two integrals appearing in (74). To achieve this, we first note that the integrand in (74) is totally symmetric in each set of variables $\{z_{\tau(i)+1}, \dots, z_{\tau(i+1)}\}$, for $i = 1, 2, \dots, m$. With this in mind we can then apply the following elementary result.

Lemma 5. *If $f(z_1, \dots, z_{\tau(i)+1}, \dots, z_{\tau(i+1)}, \dots, z_{\tau(m+1)})$ is a totally symmetric function of the variables $\{z_{\tau(i)+1}, \dots, z_{\tau(i+1)}\}$, then*

$$\begin{aligned}\prod_{l_i=\tau(i)+1}^{\tau(i+1)} \left(\int_{\Omega_{+,i}} dz_{l_i} + \int_{\Omega_{-,i}} dz_{l_i} \right) f(z_1, \dots, z_{\tau(i)+1}, \dots, z_{\tau(i+1)}, \dots, z_{\tau(m+1)}) \\ = \sum_{n_i=0}^{2q_i} \binom{2q_i}{n_i} \prod_{l_i=\tau(i)+1}^{\tau(i)+n_i} \int_{\Omega_{+,i}} dz_{l_i} \prod_{l_i=\tau(i)+n_i+1}^{\tau(i+1)} \int_{\Omega_{-,i}} dz_{l_i}\ f(z_1, \dots, z_{\tau(i)+1}, \dots, z_{\tau(i+1)}, \dots, z_{\tau(m+1)}).\end{aligned} \tag{76}$$

Proof. All terms in the expansion with n_i integrals over $\Omega_{+,i}$ can be seen to be equal by swapping the order of the integrations, permuting the arguments of f in $\{z_{\tau(i)+1}, \dots, z_{\tau(i+1)}\}$, and then relabelling the integration variables appropriately. □

Applying Lemma 5 to (74) results in

$$\begin{aligned}I_{N,m,q}(\mu) = \prod_{i=1}^{m} \sum_{n_i=0}^{2q_i} \binom{2q_i}{n_i} \prod_{l_i=\tau(i)+1}^{\tau(i)+n_i} \int_{\Omega_{+,i}} dz_{l_i} \prod_{l_i=\tau(i)+n_i+1}^{\tau(i+1)} \int_{\Omega_{-,i}} dz_{l_i} \\ \prod_{i=1}^{m} \left(\prod_{l_i=\tau(i)+1}^{\tau(i+1)} e^{-N\, S(z_{l_i}, \mu_i)}\, g_{q_i}(z_{l_i}) \right) \Delta^2_{2q_i}(z_{\tau(i)+1}, \dots, z_{\tau(i+1)}) \cdot \prod_{1 \le j < k \le m} \prod_{l_k=\tau(k)+1}^{\tau(k+1)} \prod_{l_j=\tau(j)+1}^{\tau(j+1)} (z^{\delta}_{l_k} - z^{\delta}_{l_j}) \\ + \prod_{i=1}^{m} e^{-2q_i\, N\, \mathrm{Re}\{S_i\}} \cdot O(e^{-\varepsilon N}).\end{aligned} \tag{77}$$

Now let us parameterize the integration variables in (77) so that the paths $\Omega_{\pm,i}$ become line segments, (of length 2η say), centered at $z_{+,i}$ and $z^*_{+,i}$ respectively and lying along the direction of steepest descent

$$z_{l_i} = \begin{cases} z_{+,i} + e^{i\theta_i}\, t_{l_i}, & \tau(i) + 1 \le l_i \le \tau(i) + n_i, \\ z^*_{+,i} + e^{-i\theta_i}\, t_{l_i}, & \tau(i) + n_i + 1 \le l_i \le \tau(i) + 2q_i. \end{cases} \tag{78}$$

The angles θ_i are chosen in the usual way so that with

$$a_i := \frac{S''(z_{+,i}, \mu_i)}{2} e^{2\, i\, \theta_i} = \frac{S''(z^*_{+,i}, \mu_i)}{2} e^{-2\, i\, \theta_i} \tag{79}$$

we have $a_i \in (0, \infty)$, and so

$$a_i\, t^2_{l_i} = \frac{S''(z_{+,i}, \mu_i)}{2} (z_{l_i} - z_{+,i})^2 = \frac{S''(z^*_{+,i}, \mu_i)}{2} (z_{l_i} - z^*_{+,i})^2, \tag{80}$$

with $t_{l_i} \in [-\eta, \eta]$. With this convention the integrals through $z_{+,i}$ are oriented in the negative direction, and we will compensate for this by introducing the explicit factor $(-1)^{n_i}$. Explicitly, since

$$\frac{S''(z_{+,i}, \mu_i)}{2} = \begin{cases} \dfrac{\pi}{2}\, \rho(\mu_i)\, e^{i(\mathrm{Arcsin}(\mu_i) + \pi)} & \text{Hermite,} \\ \pi\, \mu_i^2\, \rho(\mu_i)\, e^{-i\pi/2} & \text{Laguerre,} \end{cases} \tag{81}$$

we obtain

$$\theta_i = \begin{cases} \dfrac{\pi - \operatorname{Arcsin}(\mu_i)}{2} & \text{Hermite,} \\ \dfrac{\pi}{4} & \text{Laguerre,} \end{cases} \tag{82}$$

$$a_i = \begin{cases} \dfrac{\pi}{2}\,\rho(\mu_i) & \text{Hermite,} \\ \pi\,\mu_i^2\,\rho(\mu_i) & \text{Laguerre.} \end{cases} \tag{83}$$

Note that $a_i > 0$ when $\mu_i \in \operatorname{int}(\operatorname{supp}\rho)$.

In making the change of variables (78) in (77), it is convenient to introduce the the following definitions:

$$\phi_{N,i} := 4q_i\,\theta_i + 2N\operatorname{Im}\{S_i\}, \tag{84}$$

$$\varphi_i(t_{l_i}) := \sum_{k=1}^{\infty} \frac{2}{S''(z_{+,i},\mu_i)} \frac{S^{(k+2)}(z_{+,i},\mu_i)}{(k+2)!}(z_{l_i} - z_{+,i})^k \Big|_{z_{l_i} = z_{+,i} + e^{i\theta_i t_{l_i}}}, \tag{85}$$

$$\Phi_n(t) := \prod_{i=1}^{m} \prod_{l_i=\tau(i)+1}^{\tau(i)+n_i} e^{-N a_i\, t_{l_i}^2 \varphi_i(t_{l_i})} \prod_{l_i=\tau(i)+n_i+1}^{\tau(i+1)} e^{-N a_i\, t_{l_i}^2 \varphi_i^*(t_{l_i})}, \tag{86}$$

$$G_n(t) := \prod_{i=1}^{m} \prod_{l_i=\tau(i)+1}^{\tau(i+1)} g_{q_i}(z_{l_i})\Big|_{\star}, \tag{87}$$

$$D_n(t) := \prod_{i=1}^{m} \prod_{k_i=\tau(i)+n_i+1}^{\tau(i+1)} \prod_{j_i=\tau(i)+1}^{\tau(i)+n_i} (z_{k_i} - z_{j_i})^2\Big|_{\star}, \tag{88}$$

$$H_n(t) := \prod_{1\le j<k\le m} \prod_{l_k=\tau(k)+1}^{\tau(k+1)} \prod_{l_j=\tau(j)+1}^{\tau(j+1)} (z_{l_k}^{\delta} - z_{l_j}^{\delta})\Big|_{\star}, \tag{89}$$

$$F_n(t) := G_n(t) D_n(t) H_n(t), \tag{90}$$

where $\star$ on the right hand sides of (87), (88) and (89) refers to the change of variables (78), and $n := (n_1,\ldots,n_m)$. We also define the (unnormalized) integral operator

$$\mathbb{E}_{r,p}^{(i)} := \prod_{l_i=\tau(i)+r}^{\tau(i)+p} \int_{-\eta}^{\eta} dt_{l_i}\, e^{-N\,a_i\,t_{l_i}^2}\;\Delta_{p-r+1}^2(t_{\tau(i)+r},\ldots,t_{\tau(i)+p}). \tag{91}$$

Armed with these definitions, we see that since

$$\begin{aligned} &\Delta_{n_i}^2(z_{\tau(i)+1},\ldots,z_{\tau(i)+n_i})\,\Delta_{2q_i-n_i}^2(z_{\tau(i)+n_i+1},\ldots,z_{\tau(i)+2q_i}) \prod_{l_i=\tau(i)+1}^{\tau(i+1)} dz_{l_i} \\ &\qquad = e^{i4q_i(q_i-n_i)\theta_i}\Delta_{n_i}^2(t_{\tau(i)+1},\ldots,t_{\tau(i)+n_i})\;\Delta_{2q_i-n_i}^2(t_{\tau(i)+n_i+1},\ldots,t_{\tau(i)+2q_i}) \prod_{l_i=\tau(i)+1}^{\tau(i+1)} dt_{l_i}, \end{aligned} \tag{92}$$

and

$$\prod_{i=1}^{m}\prod_{l_i=\tau(i)+1}^{\tau(i+1)} e^{-NS(z_{l_i},\mu_i)} = \prod_{i=1}^{m} e^{-2q_iN\operatorname{Re}\{S_i\}} \prod_{i=1}^{m} e^{2i(q_i-n_i)N\operatorname{Im}\{S_i\}} \prod_{i=1}^{m}\prod_{l_i=\tau(i)+1}^{\tau(i+1)} e^{-Na_it_{l_i}^2}\;\Phi_n(t), \tag{93}$$

we arrive at the following more compact expression for $I_{N,m,q}(\mu)$

$$\begin{aligned} I_{N,m,q}(\mu) = &\prod_{i=1}^{m} e^{-2q_i\,N\operatorname{Re}\{S_i\}} \\ &\times \left[\prod_{i=1}^{m}\sum_{n_i=0}^{2q_i}\binom{2q_i}{n_i}(-1)^{n_i}\,e^{i(q_i-n_i)\phi_{N,i}}\,\mathbb{E}_{1,n_i}^{(i)}\mathbb{E}_{n_i+1,2q_i}^{(i)}\,\Phi_n(t)\,F_n(t) + O(e^{-\varepsilon N})\right]. \end{aligned} \tag{94}$$

The factor $(-1)^{n_i}$ results from the fact that we traversed the line through $z_{+,i}$ in the negative direction, whereas the line through $z^*_{+,i}$ is traversed in the positive direction.

To obtain an asymptotic expansion of $I_{N,m,q}(\mu)$ from (94) we proceed in direct analogy with the one-dimensional saddle point method (see e.g.[39]) and introduce the following generalization of $\Phi_n(t)$,

$$\Phi_n(t,u) := \prod_{i=1}^{m} \prod_{l_i=\tau(i)+1}^{\tau(i)+n_i} e^{u_{l_i}\varphi_i(t_{l_i})} \prod_{l_i=\tau(i)+n_i+1}^{\tau(i+1)} e^{u_{l_i}\varphi_i^*(t_{l_i})}, \tag{95}$$

where $u \in \mathbb{R}^{2|q|}$. For convenience we also introduce the function

$$Q_n(t,u) := \Phi_n(t,u)\, F_n(t), \tag{96}$$

so that with

$$u_{l_i} = -Na_i t_{l_i}^2 \quad \text{for each} \quad 1 \le l_i \le 2|q|, \tag{97}$$

we have

$$Q_n(t, \{-Na_i t_{l_i}^2\}) = \Phi_n(t)\, F_n(t). \tag{98}$$

Let us suppose for the present that u is some arbitrary fixed parameter independent of t, and consider the kth degree Taylor polynomial of $Q_n(t,u)$ as a function of $t \in \mathbb{R}^{2|q|}$

$$Q_n(t,u) = \sum_{0\le|\alpha|\le k} \frac{1}{\alpha!} \left[\frac{\partial^{\alpha_1}}{\partial t_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_{2|q|}}}{\partial t_{2|q|}^{\alpha_{2|q|}}} Q_n(t,u)\Big|_{t=0} \right] t^\alpha \,+\, O(t^\alpha)\Big|_{|\alpha|=k+1}, \tag{99}$$

where $\alpha \in \mathbb{Z}_{\ge 0}^{2|q|}$ and we use the standard notations $\alpha! = \alpha_1! \ldots \alpha_{2|q|}!$ and $t^\alpha = t_1^{\alpha_1} \ldots t_{2|q|}^{\alpha_{2|q|}}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_{2|q|}$.

If in (99) we now chose u according to (97) we obtain

$$\begin{aligned} Q_n(t, \{-Na_i t_{l_i}^2\}) = &\sum_{0\le|\alpha|\le k} \frac{1}{\alpha!} \left[\frac{\partial^{\alpha_1}}{\partial t_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_{2|q|}}}{\partial t_{2|q|}^{\alpha_{2|q|}}} \Phi_n(t,u)\, F_n(t)\Big|_{t=0} \right]\Bigg|_{\{u_{l_i}=-Na_i t_{l_i}^2\}} t^\alpha \\ &+ O(t^\alpha)\Big|_{|\alpha|=k+1}, \end{aligned} \tag{100}$$

where we emphasize that the partial derivatives with respect to t on the right hand side of (100) are performed with u fixed *before* making the substitution (97). The effect of constructing the Taylor series in this way is that when (100) is substituted into (94) and the integrations are performed, each term corresponding to a given value of $|\alpha|$ in (100) will have the same N dependence. To see this we need to consider the asymptotics of the integral operator (91) acting on a general monomial, which can be deduced simply by scaling N out of the integral. We thus deduce that

$$\prod_{i=1}^{m} \mathbb{E}^{(i)}_{1,n_i}\, \mathbb{E}^{(i)}_{n_i+1,2q_i}\; t^\alpha = O\left(N^{-\sum_{i=1}^m q_i^2 - \sum_{i=1}^m (n_i-q_i)^2 - |\alpha|/2} \right). \tag{101}$$

As a direct consequence of (101), we see that with $u_{l_i} = -a_i N t_{l_i}^2$ for each $1 \le l_i \le 2|q|$, we have

$$\prod_{i=1}^{m} \mathbb{E}^{(i)}_{1,n_i}\, \mathbb{E}^{(i)}_{n_i+1,2q_i}\; t^\alpha\, u^{\alpha'} = O\left(N^{-\sum_{i=1}^m q_i^2 - \sum_{i=1}^m (n_i-q_i)^2 - |\alpha|/2} \right) \tag{102}$$

for any $\alpha'_1, \ldots, \alpha'_{2|q|} \in \mathbb{N}$; i.e. the left hand sides of (101) and (102) have precisely the same asymptotic dependence on N. From (95) its clear that when computing the partial derivatives of $\Phi_n(t,u)\, F_n(t)$ for a given $|\alpha|$ various powers of u will appear, which after the substitution (97) will mean we need to calculate quantities of the form (102). However (101) and (102) tell us that all such quantities have the same N dependence. This would not be the case if we had just naively constructed the Taylor polynomial of $\Phi_n(t)F_n(t)$. We remark that since the symmetry of the integral operator (91) implies t^α is annihilated whenever $|\alpha|$ is odd, only integer powers of $1/N$ actually appear in the asymptotic expansion of $I_{N,m,q}(\mu)$, despite the appearance of $|\alpha|/2$ in the exponent in (102).

As a result of the expression (102) we can see that the dominant term in the asymptotic expansion of (94) occurs when both $n_i = q_i$ for all $i = 1, \ldots, m$, and $|\alpha| = 0$ in the Taylor expansion (100). In general, the coefficient of the term which is of order $1/N^k$ relative to the leading term is composed of all terms for which

$$\sum_{i=1}^{m} (n_i - q_i)^2 + \frac{|\alpha|}{2} = k. \tag{103}$$

If we are interested only in retaining the leading term the preceding arguments imply that

$$I_{N,m,q}(\mu) = \prod_{i=1}^{m} \frac{e^{-2q_i\, N\, \mathrm{Re}\{S_i\}}}{N^{q_i^2}} \cdot I_0 \left[1 + O\left(\frac{1}{N}\right)\right], \tag{104}$$

where the coefficient I_0 depends on μ and q but is independent of N. To obtain the explicit form of I_0 we first note that, since $\Phi_n(0) = 1$, we have

$$\mathbb{E}^{(i)}_{r+1,r+p}\, \Phi_n(0)\, F_n(0) = \frac{1}{N^{p^2/2}} F_n(0)\, Z_p(a_i) + O(e^{-\eta^2 a_i N}), \tag{105}$$

where

$$Z_p(a_i) := \int_{\mathbb{R}^p} d^p x\, \Delta_p^2(x) \prod_{l=1}^{p} e^{-a_i\, x_l^2}, \tag{106}$$

$$= \left(\frac{\pi}{2^{p-1}}\right)^{p/2} \frac{G(p+2)}{a_i^{p^2/2}}. \tag{107}$$

The quantity $Z_p(a_i)$ is the normalization of the joint eigenvalue pdf of the GUE and can be expressed in terms of the Selberg integral; see[8,28]. If we substitute (100) into (94) and take the leading term, which corresponds to $\sum_{i=1}^m (n_i - q_i)^2 + |\alpha|/2 = 0$, then applying (105) we find that

$$I_0 = \prod_{i=1}^{m} \binom{2q_i}{q_i} (-1)^{q_i}\, Z_{q_i}^2(a_i) \cdot F_q(0). \tag{108}$$

The explicit form of $F_q(0)$ can be obtained from the following results

$$D_q(0) = (-1)^{|q|}\, \pi^{\sum_{i=1}^m 2q_i^2} \prod_{i=1}^{m} \rho^{2q_i^2}(\mu_i), \tag{109}$$

$$G_q(0) = \begin{cases} 1, & \text{Hermite}, \\ \displaystyle\prod_{i=1}^{m} (-1)^{q_i}\, \mu_i^{2q_i^2}\, \mu_i^{-\alpha\, q_i}, & \text{Laguerre}, \end{cases} \tag{110}$$

$$H_q(0) = \prod_{1 \le j < k \le m} (\mu_k - \mu_j)^{2\, q_j\, q_k} \times \begin{cases} 2^{|q|^2 - \sum_{i=1}^m q_i^2}, & \text{Hermite}, \\ 1, & \text{Laguerre}. \end{cases} \tag{111}$$

Now let us put together what we have learned about the asymptotic behavior of $I_{N,m,q}(\mu)$ to describe the asymptotic behavior of $\mathcal{H}_{N,m,q}(\mu)$ when $q \in \mathbb{N}^m$. To this end, we substitute (109) into (108), then (108) into (104), and finally substitute (104) and (68) into (61), to obtain

$$\mathcal{H}_{N,m,q}(\mu) = N^{\sum_{i=1}^m q_i(q_i-1)} \prod_{i=1}^{m} \frac{G^2(q_i+1)}{G(2q_i+1)} 2^{-q_i^2+q_i} \pi^{2q_i^2+q_i}\, \rho^{2q_i^2}(\mu_i) \prod_{1 \le j < k \le m} (\mu_k - \mu_j)^{-4\, q_j\, q_k}$$
$$\times \prod_{i=1}^{m} e^{-2q_i\, N\, \mathrm{Re}\{S_i\}} \zeta_N^{2q_i}(\mu_i)\, a_i^{-q_i^2} \cdot G_q(0) H_q(0) \cdot h_0 \left[1 + O\left(\frac{1}{N}\right)\right]. \tag{112}$$

Using the explicit forms for $G_q(0)$ and $H_q(0)$, given by (110) and (111) respectively, one can easily verify that for both the Hermite and Laguerre cases we have the following identity

$$\prod_{i=1}^{m} e^{-2q_i\,N\,\mathrm{Re}\{S_i\}} \zeta_N^{2q_i}(\mu_i)\, a_i^{-q_i^2} \cdot G_q(0) H_q(0) \cdot h_0 \\ = \prod_{i=1}^{m} \omega_N^{-q_i}(\mu_i)\, \rho^{-q_i^2}(\mu_i)\, 2^{2q_i^2-2q_i}\, \pi^{-q_i^2-2q_i} \prod_{1\le j<k\le m} (\mu_k-\mu_j)^{2q_j\,q_k}. \quad (113)$$

Inserting the identity (113) into (112) we finally obtain

$$\begin{aligned} \mathcal{H}_{N,m,q}(\mu) = N^{\sum_{i=1}^m (q_i^2-q_i)} \prod_{i=1}^{m} [\omega_N(\mu_i)]^{-q_i} \prod_{i=1}^{m} \frac{G^2(q_i+1)}{G(2q_i+1)} (2\pi)^{q_i^2-q_i} \\ \times \prod_{1\le j<k\le m} |\mu_k-\mu_j|^{-2q_j\,q_k} \prod_{l=1}^{m} [\rho(\mu_i)]^{q_i^2} \left[1+O\left(\frac{1}{N}\right)\right], \end{aligned} \quad (114)$$

which does indeed recover (21).

We emphasize that the derivation we have presented for (114) is entirely rigorous for $q \in \mathbb{N}^m$ for any $m \in \mathbb{N}$, thus verifying the legitimacy of the log-gas procedure used in[5] for the Hermite and Laguerre cases, for such q.

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* Electronic address: garoni@ima.umn.edu; URL: `http://www.ima.umn.edu/~garoni`

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